Linearised Free-Surface Equations for Pressure Distributions

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Abstract

Important equations for velocity potential, derivatives and free-surface elevation for a region of pressure moving steadily over the surface of water are stated. Comparisons are made with equations for the same quantities in thin-ship theory.

1 On a fluid of infinite depth

The velocity potential of a unit Havelock source located at $(x, y, z) = (0, 0, \zeta)$ is [1]

$$\begin{aligned} G(x,y,z;\zeta) &= -\frac{1}{4\pi^2}\Re\int_{-\pi/2}^{\pi/2} d\theta \int_0^\infty dk\, e^{-ik(x\cos\theta + y\sin\theta)} \\ &\qquad \left[e^{-k|z-\zeta|} - \frac{k+k_0\sec^2\theta}{k-k_0\sec^2\theta}e^{k(z+\zeta)}\right] && (1) \\ &= -\frac{1}{4\pi}(\frac{1}{R_0}-\frac{1}{R}) + \frac{k_0}{2\pi^2}\Re\int_{-\pi/2}^{\pi/2} d\theta \sec^2\theta \\ &\qquad \int_0^\infty dk\, e^{-ik(x\cos\theta+y\sin\theta)+k(z+\zeta)}\frac{1}{k-k_0\sec^2\theta} && (2) \\ &= -\frac{1}{4\pi}(\frac{1}{R_0}-\frac{1}{R}) + \frac{k_0}{\pi^2}\Re\int_0^{\pi/2} d\theta \sec^2\theta \\ &\qquad \int_0^\infty dk\, e^{-ikx\cos\theta+k(z+\zeta)}\cos(ky\sin\theta)\frac{1}{k-k_0\sec^2\theta} && (3) \end{aligned}$$

where the path of k-integration passes above the pole at $k = k_0\sec^2\theta$, with $k_0 = g/U^2$ and both z and ζ non-positive. Presuming the source is located on the undisturbed free surface, i.e. $\zeta = 0$, allows the simplification (c.f. equation for $G(x, y, z)$ in [3])

$$G(x,y,z) = \frac{k_0}{2\pi^2}\Re\int_{-\pi/2}^{\pi/2} d\theta \sec^2\theta \int_0^\infty dk\, e^{-ik(x\cos\theta + y\sin\theta)+kz} \frac{1}{k - k_0\sec^2\theta}. \tag{4}$$

As stated by [4], the velocity potential induced by a unit delta-function pressure is (c.f. equation for $H(x, y, z)$ in [3])

$$\begin{aligned} H(x,y,z) & \\ &= \frac{U}{\rho g} G_x(x,y,z) \qquad (5)\\ &= -\frac{U}{\rho g}\frac{k_0}{2\pi^2}\Re i\int_{-\pi/2}^{\pi/2} d\theta \sec\theta \int_0^\infty dk\, k e^{-ik(x\cos\theta + y\sin\theta)+kz} \frac{1}{k - k_0\sec^2\theta}. \qquad (6) \end{aligned}$$

Thus, the velocity potential of a distribution of pressure $P(x, y)$ over a region B is

$$\begin{aligned} \phi(x,y,z) & \\ &= \frac{U}{\rho g}\iint_B d\xi d\eta P(\xi,\eta) G_x(x-\xi, y-\eta, z) \qquad (7)\\ &= -\frac{U}{\rho g}\frac{k_0}{2\pi^2}\Re i\int_{-\pi/2}^{\pi/2} d\theta \sec\theta \\ &\quad \int_0^\infty dk\, k e^{-ik(x\cos\theta + y\sin\theta)+kz} \frac{1}{k - k_0\sec^2\theta}(P_P(\theta,k) + iQ_P(\theta,k)) \qquad (8) \end{aligned}$$

where

$$P_P(\theta,k) + iQ_P(\theta,k) = \iint_B d\xi d\eta P(\xi,\eta) e^{ik(\xi\cos\theta + \eta\sin\theta)}. \tag{9}$$

This can be compared with the potential due to a thin ship [2]

$$\begin{aligned} \phi(x,y,z) & \\ &= 2U\iint_R d\xi d\zeta Y_x(\xi,\zeta) G(x-\xi, y, z; \zeta) \qquad (10)\\ &= -2U\left[\frac{-1}{4\pi}\left(\frac{1}{R_0} - \frac{1}{R}\right) + \frac{k_0}{2\pi^2}\Re i\int_{-\pi/2}^{\pi/2} d\theta \sec\theta \right.\\ &\quad \left. \int_0^\infty dk k e^{-ik(x\cos\theta + y\sin\theta)+kz} \frac{1}{k - k_0\sec^2\theta}(P_T(\theta,k) + iQ_T(\theta,k))\right] \qquad (11) \end{aligned}$$

where

$$
\begin{aligned}
P_T(\theta, k) &+ iQ_T(\theta, k) \\
&= \frac{1}{-ik\cos\theta} \iint_R d\xi d\zeta \; Y_x(\xi, \zeta) \; e^{ik\xi\cos\theta + k\zeta} && (12) \\
&= \iint_R d\xi d\zeta \; Y(\xi, \zeta) \; e^{ik\xi\cos\theta + k\zeta} - \frac{1}{ik\cos\theta} e^{ik\xi_S\cos\theta} \int Y(\xi_S, \zeta) e^{k\zeta} d\zeta. && (13)
\end{aligned}
$$

Note that if (11) is evaluated at $z = 0$, then it is identical to (8) except that the thin-ship scale factor of $2U$ is replaced by the pressure-distribution scale factor of $U/(\rho g)$.

Turning attention back to a pressure distribution and considering the far-field flow, define

$$
\phi^F(x, y, z) = -2\pi i \,\mathrm{Res}\, \phi(x, y, z) \tag{14}
$$

where Res is the residue at the pole $k = k_0 \sec^2\theta$. Thus

$$
\begin{aligned}
\phi^F(x, y, z) &= -\frac{U}{\rho g}\frac{k_0}{\pi} \Re \int_{-\pi/2}^{\pi/2} d\theta \sec\theta k e^{-ik(x\cos\theta + y\sin\theta) + kz} \\
&\quad (P_P^F(x, \theta) + iQ_P^F(x, \theta)) && (15)
\end{aligned}
$$

where

$$
P_P^F(x, \theta) + iQ_P^F(x, \theta) = \iint_B d\xi d\eta P(\xi, \eta) e^{ik(\xi\cos\theta + \eta\sin\theta)} H(x - \xi), \tag{16}
$$

$k = k_0 \sec^2\theta$ and the Heaviside step function has been introduced so that only that portion of the ship ahead of the point of consideration contributes to the waves. Its derivatives are:

$$
\begin{aligned}
\phi_x^F(x, y, z) &= \frac{U}{\rho g}\frac{k_0}{\pi} \Re i \int_{-\pi/2}^{\pi/2} d\theta k^2 e^{-ik(x\cos\theta + y\sin\theta) + kz} \\
&\quad (P_P^F(x, \theta) + iQ_P^F(x, \theta)) \\
&\quad -\frac{U}{\rho g}\frac{k_0}{\pi} \Re \int_{-\pi/2}^{\pi/2} d\theta \sec\theta k e^{-ik(x\cos\theta + y\sin\theta) + kz} \\
&\quad \frac{\partial}{\partial x}(P_P^F(x, \theta) + iQ_P^F(x, \theta)), && (17)
\end{aligned}
$$

$$
\begin{aligned}
\phi_y^F(x, y, z) &= \frac{U}{\rho g}\frac{k_0}{\pi} \Re i \int_{-\pi/2}^{\pi/2} d\theta \sec\theta \sin\theta k^2 e^{-ik(x\cos\theta + y\sin\theta) + kz} \\
&\quad (P_P^F(x, \theta) + iQ_P^F(x, \theta)), && (18)
\end{aligned}
$$

and

$$\phi_z^F(x,y,z) = -\frac{U}{\rho g}\frac{k_0}{\pi}\Re\int_{-\pi/2}^{\pi/2} d\theta \sec\theta k^2 e^{-ik(x\cos\theta+y\sin\theta)+kz} (P_P^F(x,\theta)+iQ_P^F(x,\theta)), \tag{19}$$

where

$$\frac{\partial}{\partial x}(P_P^F(x,\theta)+iQ_P^F(x,\theta)) = H(x-x_b)(1-H(x-x_s))\int d\eta P(x,\eta)e^{ik(x\cos\theta+\eta\sin\theta)}. \tag{20}$$

The far-field free-surface elevation is

$$Z^F(x,y) = -\frac{U}{g}\phi_x(x,y,0) \tag{21}$$

$$= -\frac{1}{\rho g}\frac{1}{\pi}\Re i\int_{-\pi/2}^{\pi/2} d\theta k^2 e^{-ik(x\cos\theta+y\sin\theta)} (P_P^F(x,\theta)+iQ_P^F(x,\theta)) + \frac{1}{\rho g}\frac{1}{\pi}\Re\int_{-\pi/2}^{\pi/2} d\theta \sec\theta k e^{-ik(x\cos\theta+y\sin\theta)} \frac{\partial}{\partial x}(P_P^F(x,\theta)+iQ_P^F(x,\theta)). \tag{22}$$

The wave resistance is given by the integral [3]

$$R = \frac{\pi}{2}\rho U^2\int_{-\pi/2}^{\pi/2} d\theta |A(\theta)|^2\cos^3\theta \tag{23}$$

where $A(\theta)$ is defined by

$$Z^F(x,y) = \Re\int_{-\pi/2}^{\pi/2} d\theta A(\theta)e^{-ik(x\cos\theta+y\sin\theta)}, \tag{24}$$

the restriction being that $x > x_s$ so that the second term in Z^F doesn't contribute. Here

$$A(\theta) = \frac{-i}{\pi\rho g}k^2(P_P^F(x,\theta)+iQ_P^F(x,\theta)) \tag{25}$$

so

$$R = \frac{1}{2\pi\rho g k_0}\int_{-\pi/2}^{\pi/2} d\theta k^4 |P_P^F(x,\theta)+iQ_P^F(x,\theta)|^2\cos^3\theta. \tag{26}$$

Simplification yields

$$R = \frac{k_0^3}{\pi\rho g}\int_0^{\pi/2} d\theta \sec^5\theta |P_P^F(x,\theta) + iQ_P^F(x,\theta)|^2. \tag{27}$$

In each case except the wave resistance, the corresponding thin-ship formula can be obtained by replacing a factor of $U/(\rho g)$ with $2U$ and $P_P^F(x,\theta) + iQ_P^F(x,\theta)$ with $P_T^F(x,\theta) + iQ_T^F(x,\theta)$ where

$$\begin{aligned} & P_T^F(x,\theta) + iQ_T^F(x,\theta) \\ &= \frac{1}{-ik\cos\theta}\iint_R d\xi d\zeta\ Y_x(\xi,\zeta)\ e^{ik\xi\cos\theta + k\zeta} H(x-\xi) \qquad (28) \\ &= \iint_R d\xi d\zeta\ Y(\xi,\zeta)\ e^{ik\xi\cos\theta+k\zeta} H(x-\xi) \\ &\quad - \frac{1}{ik\cos\theta} H(x-x_b) e^{ikx^*\cos\theta}\int Y(x^*,\zeta) e^{k\zeta} d\zeta \qquad (29) \end{aligned}$$

and $k = k_0\sec^2\theta$ and $x^* = \min(x, x_s)$, the offset at the bow being taken as zero. Its x-derivative is

$$\begin{aligned} & \frac{\partial}{\partial x}(P_T^F(x,\theta) + iQ_T^F(x,\theta)) \\ &= \frac{1}{-ik\cos\theta}\iint_R d\xi d\zeta\ Y_x(\xi,\zeta)\ e^{ik\xi\cos\theta+k\zeta}\delta(x-\xi) \qquad (30) \\ &= H(x-x_b)(1-H(x-x_s))\frac{1}{-ik\cos\theta}\int d\zeta Y_x(x,\zeta) e^{ikx\cos\theta+k\zeta} \qquad (31) \end{aligned}$$

but importantly the second term in the expression for $\phi^F(x,y,z)$ is then zero. For the wave resistance, a factor of $(U/(\rho g))^2$ must be replaced by $(2U)^2$ and $P_P^F(x,\theta) + iQ_P^F(x,\theta)$ must be replaced by $P_T^F(x,\theta) + iQ_T^F(x,\theta)$.

2 On a fluid of finite depth

The velocity potential of a unit Havelock source located at $(x,y,z) = (0,0,0)$ in a fluid of depth h is

$$\begin{aligned} G(x,y,z) &= \frac{k_0}{2\pi^2}\Re\int_{-\pi/2}^{\pi/2} d\theta \sec^2\theta \\ &\qquad \int_0^\infty dk\, e^{-ik(x\cos\theta + y\sin\theta)+kz}\frac{\cosh k(z+h)\,\mathrm{sech}\, kh}{k - k_0\sec^2\theta\tanh kh} \qquad (32) \end{aligned}$$

where the path of k-integration passes above the pole at $k = k_0 \sec^2\theta \tanh kh$, with $k_0 = g/U^2$ and ζ non-positive.

The velocity potential induced by a unit delta-function pressure is then

$$
\begin{aligned}
H(x,y,z) & \\
&= \frac{U}{\rho g} G_x(x,y,z) \qquad (33)\\
&= -\frac{U}{\rho g}\frac{k_0}{2\pi^2}\Re i \int_{-\pi/2}^{\pi/2} d\theta \sec\theta \\
&\quad \int_0^\infty dk\, k e^{-ik(x\cos\theta + y\sin\theta)+kz} \frac{\cosh k(z+h)\,\mathrm{sech}\, kh}{k - k_0\sec^2\theta\tanh kh}. \qquad (34)
\end{aligned}
$$

Thus, the velocity potential of a distribution of pressure $P(x,y)$ over a region B is [4, page 599]

$$
\begin{aligned}
\phi(x,y,z) & \\
&= \frac{U}{\rho g}\iint_B d\xi d\eta P(\xi,\eta) G_x(x-\xi, y-\eta, z) \qquad (35)\\
&= -\frac{U}{\rho g}\frac{k_0}{2\pi^2}\Re i \int_{-\pi/2}^{\pi/2} d\theta \sec\theta \\
&\quad \int_0^\infty dk\, k e^{-ik(x\cos\theta + y\sin\theta)+kz} \frac{\cosh k(z+h)\,\mathrm{sech}\, kh}{k - k_0\sec^2\theta\tanh kh} \\
&\quad (P_P(\theta,k) + iQ_P(\theta,k)) \qquad (36)
\end{aligned}
$$

where

$$P_P(\theta,k) + iQ_P(\theta,k) = \iint_B d\xi d\eta P(\xi,\eta) e^{ik(\xi\cos\theta + \eta\sin\theta)} \qquad (37)$$

as before. Note that these formulae for the total potential differ from the infinite-depth case only by replacing a factor of

$$\frac{1}{k - k_0\sec^2\theta} \qquad (38)$$

with

$$\frac{\cosh k(z+h)\,\mathrm{sech}\, kh}{k - k_0\sec^2\theta\tanh kh}. \qquad (39)$$

For the far-field flow, define

$$\phi^F(x,y,z) = -2\pi i \,\mathrm{Res}\, \phi(x,y,z) \qquad (40)$$

where Res is the residue at the pole $k = k_0 \sec^2\theta \tanh kh$. Thus

$$\begin{aligned}\phi^F(x,y,z) &= -\frac{U}{\rho g}\frac{k_0}{\pi}\Re\int_{-\pi/2}^{\pi/2} d\theta \sec\theta k e^{-ik(x\cos\theta+y\sin\theta)+kz} \\ &\quad \frac{\cosh k(z+h)\,\mathrm{sech}\,kh}{1-k_0h\sec^2\theta\,\mathrm{sech}^2 kh}(P_P^F(x,\theta)+iQ_P^F(x,\theta)) \qquad (41)\end{aligned}$$

where

$$P_P^F(x,\theta)+iQ_P^F(x,\theta) = \iint_B d\xi d\eta P(\xi,\eta)e^{ik(\xi\cos\theta+\eta\sin\theta)}H(x-\xi), \qquad (42)$$

$k = k_0\sec^2\theta\tanh kh$ and the Heaviside step function has been introduced so that only that portion of the ship ahead of the point of consideration contributes to the waves. Its derivatives are:

$$\begin{aligned}\phi_x^F(x,y,z) &= \frac{U}{\rho g}\frac{k_0}{\pi}\Re i\int_{-\pi/2}^{\pi/2} d\theta k^2 e^{-ik(x\cos\theta+y\sin\theta)+kz} \\ &\quad \frac{\cosh k(z+h)\,\mathrm{sech}\,kh}{1-k_0h\sec^2\theta\,\mathrm{sech}^2 kh}(P_P^F(x,\theta)+iQ_P^F(x,\theta)) \\ &\quad -\frac{U}{\rho g}\frac{k_0}{\pi}\Re\int_{-\pi/2}^{\pi/2} d\theta\sec\theta k e^{-ik(x\cos\theta+y\sin\theta)+kz} \\ &\quad \frac{\cosh k(z+h)\,\mathrm{sech}\,kh}{1-k_0h\sec^2\theta\,\mathrm{sech}^2 kh}\frac{\partial}{\partial x}(P_P^F(x,\theta)+iQ_P^F(x,\theta)), \qquad (43)\end{aligned}$$

$$\begin{aligned}\phi_y^F(x,y,z) &= \frac{U}{\rho g}\frac{k_0}{\pi}\Re i\int_{-\pi/2}^{\pi/2} d\theta\sec\theta\sin\theta k^2 e^{-ik(x\cos\theta+y\sin\theta)+kz} \\ &\quad \frac{\cosh k(z+h)\,\mathrm{sech}\,kh}{1-k_0h\sec^2\theta\,\mathrm{sech}^2 kh}(P_P^F(x,\theta)+iQ_P^F(x,\theta)), \qquad (44)\end{aligned}$$

and

$$\begin{aligned}\phi_z^F(x,y,z) &= -\frac{U}{\rho g}\frac{k_0}{\pi}\Re\int_{-\pi/2}^{\pi/2} d\theta\sec\theta k^2 e^{-ik(x\cos\theta+y\sin\theta)+kz} \\ &\quad \frac{\cosh k(z+h)\,\mathrm{sech}\,kh}{1-k_0h\sec^2\theta\,\mathrm{sech}^2 kh}(P_P^F(x,\theta)+iQ_P^F(x,\theta)), \qquad (45)\end{aligned}$$

where

$$\frac{\partial}{\partial x}(P_P^F(x,\theta)+iQ_P^F(x,\theta)) = H(x-x_b)(1-H(x-x_s))\int d\eta P(x,\eta)e^{ik(x\cos\theta+\eta\sin\theta)}. \qquad (46)$$

The far-field free-surface elevation is

$$
\begin{aligned}
Z^F(x,y) &= -\frac{U}{g}\phi_x(x,y,0) && (47)\\
&= -\frac{1}{\rho g}\frac{1}{\pi}\Re i\int_{-\pi/2}^{\pi/2} d\theta k^2 e^{-ik(x\cos\theta+y\sin\theta)}\\
&\quad \frac{1}{1-k_0h\sec^2\theta\,\mathrm{sech}^2 kh}(P_P^F(x,\theta)+iQ_P^F(x,\theta))\\
&\quad +\frac{1}{\rho g}\frac{1}{\pi}\Re\int_{-\pi/2}^{\pi/2} d\theta\sec\theta k e^{-ik(x\cos\theta+y\sin\theta)}\\
&\quad \frac{\cosh k(z+h)\,\mathrm{sech}\,kh}{1-k_0h\sec^2\theta\,\mathrm{sech}^2 kh}\frac{\partial}{\partial x}(P_P^F(x,\theta)+iQ_P^F(x,\theta)). && (48)
\end{aligned}
$$

Note that these formulae for the far-field potential and its derivatives differ from the infinite-depth case only by using the finite-depth value of k and introducing a factor of

$$\frac{\cosh k(z+h)\,\mathrm{sech}\,kh}{1-k_0h\sec^2\theta\,\mathrm{sech}^2 kh} \tag{49}$$

which, for the free-surface elevation where $z=0$, simplifies to

$$\frac{1}{1-k_0h\sec^2\theta\,\mathrm{sech}^2 kh}. \tag{50}$$

The wave resistance is given by the integral [3]

$$R=\frac{\pi}{2}\rho U^2\int_{-\pi/2}^{\pi/2} d\theta|A(\theta)|^2\cos^3\theta \tag{51}$$

where $A(\theta)$ is defined by

$$Z^F(x,y)=\Re\int_{-\pi/2}^{\pi/2} d\theta A(\theta)e^{-ik(x\cos\theta+y\sin\theta)}, \tag{52}$$

the restriction being that $x>x_s$ so that the second term in Z^F doesn't contribute. Here

$$A(\theta)=\frac{-i}{\pi\rho g}k^2\frac{1}{1-k_0h\sec^2\theta\,\mathrm{sech}^2 kh}(P_P^F(x,\theta)+iQ_P^F(x,\theta)) \tag{53}$$

so

$$R = \frac{1}{2\pi\rho g k_0} \int_{-\pi/2}^{\pi/2} d\theta k^4 \frac{1}{(1 - k_0 h \sec^2\theta \,\mathrm{sech}^2 kh)^2} |P_P^F(x,\theta) + iQ_P^F(x,\theta)|^2 \cos^3\theta. \tag{54}$$

This differs for the formula for the infinite-depth case only by using the finite-depth value of k and introducing a factor of

$$\frac{1}{(1 - k_0 h \sec^2\theta \,\mathrm{sech}^2 kh)^2}. \tag{55}$$

Simplification yields

$$R = \frac{k_0^3}{\pi\rho g} \int_0^{\pi/2} d\theta \sec^5\theta \frac{\tanh^4 kh}{(1 - k_0 h \sec^2\theta \,\mathrm{sech}^2 kh)^2} |P_P^F(x,\theta) + iQ_P^F(x,\theta)|^2. \tag{56}$$

3 Damping due to eddy viscosity

It is possible to include viscosity in the far-field calculations by introducing to the integrands the multiplicative factor

$$\exp(-2\nu k^2 \frac{x}{U}) \tag{57}$$

where ν is the eddy viscosity, and $k = k_0 \sec^2\theta$ for infinite-depth fluid and $k = k_0 \sec^2\theta \tanh kh$ for finite-depth fluid.

This factor has only an empirical basis, and is mathematically flawed. It depends unreasonably upon the location of the origin. If included in the formulae for particle velocities and surface elevation, then these quantities are no longer partial derivatives of the velocity potential.

References

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