SEA WAVE PATTERN EVALUATION — PRESSURE DISTRIBUTIONS:

Mathematical Formulation

D.C. Scullen and E.O. Tuck

Scullen & Tuck Pty Ltd
PO Box 237 Rundle Mall, Adelaide, South Australia 5000

July 31, 2001

Abstract

The computer program `SWPE-PD` provides, in an efficient and accurate manner, the wave elevation and flow field created by a travelling pressure distribution, moving steadily forward in still water of constant depth. This report details the mathematical formulation of the problem and the manner in which it is solved. Important equations for velocity potential, derivatives and free-surface elevation in both the far and local fields are stated.

1 Introduction

The program SWPE-PD (Sea Wave Pattern Evaluation — Pressure Distributions) is a computational procedure for determining the shape $z = Z(x, y)$ of the disturbance to the otherwise-plane free surface $z = 0$ of a body of water, due to passage at uniform speed U of a pressure distribution having a small prescribed excess $p(x, y)$ over atmospheric. The fully three-dimensional wave field can be determined with high accuracy everywhere, namely around, under, near to, or far from the pressure disturbance.

There are obvious direct applications, to travelling meteorological disturbances or to hovercraft, for example. In addition, there are indirect or inverse potential applications, where one may seek to determine the pressure distribution $p(x, y)$ in order to produce a specified disturbance $Z(x, y)$ within some portion of the plane $z = 0$. This is the topic of "flat-ship" theory, and is related to planing. Although we shall not consider such inverse problems here, the algorithms produced for the direct problem are intended to serve as the basis for solution of the inverse problem.

The analytic relationship between the input $p(x, y)$ and the output $Z(x, y)$ is well known (see e.g. Wehausen and Laitone [15]), and involves a quadruple integration, namely integrals with respect to both spatial co-ordinates and with respect to the wavenumber and direction of the waves. The fact that this task involves four numerical integrations, some of which are inherently of significant computational complexity (specifically for example due to the need to cope with rapidly-oscillating integrands), has meant that only a few detailed three-dimensional computations have been attempted in the past, at least in the near field of the disturbance. Some examples are in [1], [2] and [3].

It is possible to decompose the flow and hence the wave elevation Z into the sum of two components that represent respectively the (wave-less) local and (wave-like) far-field behaviour. Although the terminology "local" and "far" will be used to describe these components, each of them must be evaluated at every point of the desired field of observation. It is true, however, that the local component decays quite rapidly as we move away from the disturbance.

The local component, which remains a quadruple integral, can be handled efficiently in a manner described by Newman [5], which effectively eliminates the wavenumber and wave-direction integrals, so reducing its complexity to that of a double spatial integral. The far-field component, by comparison, is a triple integral which can be handled efficiently for computations over a large grid by first performing a once-only double spatial integration, followed by a single wave-direction integration which must be performed separately for each required field point. This last integration (with rapidly-varying integrand) is difficult and computationally expensive, but we use acceleration techniques that have been tested in other contexts, to produce accurate results in reasonable computer times.

2 Pressure points and surface sources

We now state the important equations for the velocity potential, its derivatives and the free-surface elevation for a region of pressure moving steadily over the surface of water.

The velocity potential for the flow induced by a unit delta-function pressure [15, page 598] exerted on the free surface (at the origin) of a stream U of finite depth h is proportional to G_x, where $G(x, y, z)$ is the velocity potential for a unit Havelock source located at the free surface [15, page 484], namely

$$G(x,y,z) = \frac{k_0}{2\pi^2}\Re\int_{-\pi/2}^{\pi/2} d\theta \sec^2\theta \int_0^\infty dk\, e^{-ik(x\cos\theta+y\sin\theta)}\tfrac{\cosh k(z+h)\,\mathrm{sech}\,kh}{k-k_1}. \quad (1)$$

Here, the path of k-integration passes above the pole at $k = k_1 = k_0 \sec^2\theta$ $\tanh kh$, with $k_0 = g/U^2$ and z non-positive.

We note immediately that for the case of an infinite-depth fluid (i.e., where $h \to \infty$)

$$\cosh k(z+h)\,\mathrm{sech}\,kh \to e^{kz} \quad (2)$$

and

$$k_0 \sec^2\theta \tanh kh \to k_0 \sec^2\theta. \quad (3)$$

One can obtain equivalent infinite-depth equations from those that are to follow by making these simple substitutions. For example, the velocity potential of a unit Havelock source in water of infinite depth is

$$G(x,y,z) = \frac{k_0}{2\pi^2}\Re\int_{-\pi/2}^{\pi/2} d\theta \sec^2\theta \int_0^\infty dk\, e^{-ik(x\cos\theta+y\sin\theta)+kz}\frac{1}{k-k_0\sec^2\theta}. \quad (4)$$

Returning attention to the more general finite-depth case, the velocity potential of a distribution of pressure $p(x, y)$ over a region B of the plane

$z = 0$ is then [15, page 599]

$$\phi(x,y,z) = \frac{U}{\rho g}\iint_B d\xi d\eta\, p(\xi,\eta) G_x(x-\xi, y-\eta, z) \tag{5}$$

$$= -\frac{U}{\rho g}\frac{k_0}{2\pi^2}\Re i \int_{-\pi/2}^{\pi/2} d\theta \sec\theta \int_0^\infty dk\, k e^{-ik(x\cos\theta + y\sin\theta)} \frac{\cosh k(z+h)\,\mathrm{sech}\, kh}{k-k_1}$$
$$\iint_B d\xi d\eta\, p(\xi,\eta) e^{ik(\xi\cos\theta+\eta\sin\theta)}. \tag{6}$$

Thus, for any given pressure $p(x,y)$, the potential $\phi(x,y,z)$ is given by a quadruple integral (with respect to ξ, η, k, θ), and one can expect its evaluation to be computationally expensive.

To obtain the actual wave elevation $z = Z(x,y)$ we use the dynamic condition $p/\rho + gZ + U\phi_x = 0$, to give

$$Z(x,y) = -\frac{p(x,y)}{\rho g} - \frac{1}{\rho g k_0}\iint_B d\xi d\eta\; p(\xi,\eta)\; G_{xx}(x-\xi, y-\eta, 0) \tag{7}$$

where the first term is the hydrostatic displacement due to the given pressure $p(x,y)$.

2.1 The far field

The local-far decomposition for the actual Green's function can be written

$$G(x,y,z) = G^L(x,y,z) + G^F(x,y,z) \tag{8}$$

where (with the path of integration now diverted below the pole)

$$G^L(x,y,z) = \frac{k_0}{2\pi^2}\Re \int_{-\pi/2}^{\pi/2} d\theta \sec^2\theta \int_0^\infty dk\, e^{-ik(|x|\cos\theta + y\sin\theta)} \frac{\cosh k(z+h)\,\mathrm{sech}\, kh}{k-k_1} \tag{9}$$

and

$$G^F(x,y,z) = -\frac{k_0}{\pi} H(x) \Re i \int_{-\pi/2}^{\pi/2} d\theta \sec^2\theta e^{-ik_1(x\cos\theta + y\sin\theta)} \frac{\cosh k_1(z+h)\,\mathrm{sech}\, k_1 h}{1 - k_0 h \sec^2\theta\, \mathrm{sech}^2 k_1 h}. \tag{10}$$

In the representation (10) for the far-field potential G^F, the quantity $H(x)$ is the Heaviside unit step function, i.e. $H = 0$ for $x < 0$ and $H = 1$ for $x > 0$. Thus the potential G^F vanishes identically ahead of the point disturbance, and for pressure points distributed over a region B, at each fixed x value only those pressure points located at points (ξ, η) ahead of the point of observation (x, y) can contribute.

Writing $\phi = \phi^L + \phi^F$ to represent the local and far-field components of the potential, we have

$$\phi^L(x, y, z) = \frac{U}{\rho g} \iint_B d\xi d\eta \; p(\xi, \eta) \; G_x^L(x - \xi, y - \eta, z) \tag{11}$$

and

$$\phi^F(x, y, z) = \\ -\frac{U}{\rho g}\frac{k_0}{\pi}\Re \int_{-\pi/2}^{\pi/2} d\theta \sec\theta k_1 e^{-ik_1(x\cos\theta + y\sin\theta)} \tfrac{\cosh k_1(z+h)\,\mathrm{sech}\, k_1 h}{1 - k_0 h \sec^2\theta\,\mathrm{sech}^2 k_1 h}(P(x, \theta) + iQ(x, \theta)) \tag{12}$$

where

$$P(x, \theta) + iQ(x, \theta) = \iint_B d\xi d\eta \; p(\xi, \eta) e^{ik_1(\xi\cos\theta + \eta\sin\theta)} H(x - \xi). \tag{13}$$

Note that

$$\phi^F(x, y, z) = -2\pi i \,\mathrm{Res}\, \phi(x, y, z) \tag{14}$$

where Res is the residue at that pole.

Note also that for a fluid of infinite depth

$$\frac{1}{1 - k_0 h \sec^2\theta\,\mathrm{sech}^2 k_1 h} \to 1 \tag{15}$$

and one may again obtain the equivalent infinite-depth equations by making this simple substitution.

In general, $P = P(x, \theta)$ and $Q = Q(x, \theta)$, i.e. these functions must be computed and stored separately not just for each wave-propagation angle θ, but also for each value of the co-ordinate x (but not y). However, as is clear from the properties of the Heaviside function $H(x - \xi)$, they vanish entirely for all points ahead of the pressure region B and are independent of x for all points astern of B.

The derivatives of the far-field potential are:

$$\phi_x^F(x,y,z) = \frac{U}{\rho g}\frac{k_0}{\pi}\Re i\int_{-\pi/2}^{\pi/2} d\theta\, k_1^2 e^{-ik_1(x\cos\theta+y\sin\theta)}\tfrac{\cosh k_1(z+h)\,\mathrm{sech}\,k_1 h}{1-k_0 h\sec^2\theta\,\mathrm{sech}^2 k_1 h}(P+iQ) - \frac{U}{\rho g}\frac{k_0}{\pi}\Re\int_{-\pi/2}^{\pi/2} d\theta\sec\theta k_1 e^{-ik_1(x\cos\theta+y\sin\theta)}\tfrac{\cosh k_1(z+h)\,\mathrm{sech}\,k_1 h}{1-k_0 h\sec^2\theta\,\mathrm{sech}^2 k_1 h}\tfrac{\partial}{\partial x}(P+iQ), \tag{16}$$

$$\phi_y^F(x,y,z) = \frac{U}{\rho g}\frac{k_0}{\pi}\Re i\int_{-\pi/2}^{\pi/2} d\theta\sec\theta\sin\theta k_1^2 e^{-ik_1(x\cos\theta+y\sin\theta)}\tfrac{\cosh k_1(z+h)\,\mathrm{sech}\,k_1 h}{1-k_0 h\sec^2\theta\,\mathrm{sech}^2 k_1 h}(P+iQ), \tag{17}$$

and

$$\phi_z^F(x,y,z) = -\frac{U}{\rho g}\frac{k_0}{\pi}\Re\int_{-\pi/2}^{\pi/2} d\theta\sec\theta k_1^2 e^{-ik_1(x\cos\theta+y\sin\theta)}\tfrac{\cosh k_1(z+h)\,\mathrm{sech}\,k_1 h}{1-k_0 h\sec^2\theta\,\mathrm{sech}^2 k_1 h}(P+iQ), \tag{18}$$

where

$$\frac{\partial}{\partial x}(P(x,\theta)+iQ(x,\theta)) = H(x-x_b)(1-H(x-x_s))\int d\eta\, p(x,\eta)e^{ik_1(x\cos\theta+\eta\sin\theta)}. \tag{19}$$

The far-field free-surface elevation is

$$Z^F(x,y) = -\frac{U}{g}\phi_x^F(x,y,0) \tag{20}$$

$$= -\frac{1}{\rho g}\frac{1}{\pi}\Re i\int_{-\pi/2}^{\pi/2} d\theta\, k_1^2 e^{-ik_1(x\cos\theta+y\sin\theta)}\frac{1}{1-k_0 h\sec^2\theta\,\mathrm{sech}^2 k_1 h}(P+iQ) + \frac{1}{\rho g}\frac{1}{\pi}\Re\int_{-\pi/2}^{\pi/2} d\theta\sec\theta k_1 e^{-ik_1(x\cos\theta+y\sin\theta)}\frac{1}{1-k_0 h\sec^2\theta\,\mathrm{sech}^2 k_1 h}\frac{\partial}{\partial x}(P+iQ), \tag{21}$$

and the wave resistance is

$$R = \frac{k_0^3}{\pi\rho g}\int_0^{\pi/2} d\theta\sec^5\theta\frac{\tanh^4 k_1 h}{\left(1-k_0 h\sec^2\theta\,\mathrm{sech}^2 k_1 h\right)^2}|P+iQ|^2. \tag{22}$$

When differentiating the far-field potential with respect to x (as is required for both the x-velocity and the free-surface elevation) care must be taken to ensure that the dependence of $P + iQ$ on x is accounted for. This dependence creates, for a pressure patch with non-zero pressure at the leading or trailing edge, a discontinuity in the far-field elevation at and to the sides of that edge. However, a discontinuity of equal magnitude but opposite sign occurs in the local-field elevation, so that the resulting total free surface is continuous.

In previous work on thin ships [9][10][11][12][13][14], we have developed techniques for fast and accurate evaluation of integrals of rapidly oscillating functions similar to (12). We have found that Filon quadrature accurately captures the oscillatory nature of (13) and, when combined in the far field with the pseudo-stationary-phase algorithm described by Tuck *et al* [8], enables accurate evaluation of (12) and its derivatives everywhere in reasonable computer times.

2.2 Local field

The infinite-fluid-depth equivalent of G^L as defined in equation (9) is

$$G^L(x, y, z) = \frac{k_0}{2\pi^2}\Re \int_{-\pi/2}^{\pi/2} d\theta \sec^2\theta \int_0^\infty dk \frac{e^{-ik(|x|\cos\theta + y\sin\theta)+kz}}{k - k_0\sec^2\theta} \tag{23}$$

and is that for which rapidly-computable 6-figure accurate polynomial approximations due to Newman [5] are available. Newman in fact gave tables of values and associated formulae only for G^L itself, but it is possible to determine corresponding formulae for the first and second derivatives of G^L that are required in the present application. However, both his and our analysis is restricted to the case of an infinite-depth fluid.

There is no difficulty in direct computation of the local component at points not close to the pressure patch. Although the local component is not wave-like and does not require integration of rapidly-varying integrands, it does require integration of a distribution of pressure points, and the integrand is highly singular near the location of each such pressure point. For the surface elevation, the integral involves G^L_{xx} which is both singular and an even function of x, and thus the integral is divergent; formally a Hadamard interpretation is required. This singularity must be treated carefully to ensure good accuracy.

For example, consider the x-component of the local-field velocity, due to a rectangular region $|x| < a$, $|y| < b$ of applied pressure $p(x, y)$,

$$\frac{\rho g}{U}\phi_x^L(x,y,z) = \int_{-a}^{a}\int_{-b}^{b} p(\xi,\eta) G_{xx}^L(x-\xi, y-\eta, z)\, d\eta\, d\xi \tag{24}$$

$$\begin{aligned}
&= \int_{-b}^{b}\int_{-a}^{a} \Big(p(\xi,\eta) - p(x,\eta)\Big) G_{xx}^L(x-\xi, y-\eta, z)\, d\xi\, d\eta \\
&- \int_{-b}^{b} \Big(p(x,\eta) - p(x,y)\Big)\Big(G_x^L(x-a, y-\eta, z) - G_x^L(x+a, y-\eta, z)\Big)\, d\eta \\
&- p(x,y)\int_{-b}^{b} G_x^L(x-a, y-\eta, z) - F_y(x-a, y-\eta)\, d\eta \\
&+ p(x,y)\int_{-b}^{b} G_x^L(x+a, y-\eta, z) - F_y(x+a, y-\eta)\, d\eta \\
&- p(x,y)\int_{-b}^{b} F_y(x-a, y-\eta)\, d\eta + p(x,y)\int_{-b}^{b} F_y(x+a, y-\eta)\, d\eta.
\end{aligned} \tag{25}$$

Here,

$$F_y(x,y) = \operatorname{sgn}(x)\frac{k_0^2}{4\pi}\left[\log\frac{k_0(r+|x|)}{4} + \frac{|x|}{r+|x|} + \gamma\right] \tag{26}$$

(with γ denoting Euler's constant and $r = \sqrt{x^2+y^2}$) models the small-argument behaviour of $G_x^L(x, y, 0)$. That is, $G_x^L(x, y, z) \to F_y(x, y)$ as $(x, y, z) \to (0, 0, 0)$. This can be found by differentiating the local-field component of Newman's equation (11) [5] and completing an analysis similar to that of his Section 3. It is essentially the small-k_0 limit of G_x, and as such is closely related to the xxy-antiderivative of the kernel of the aerodynamic lifting-surface equation given by Tuck ([7], equation (1.5)).

Note that whereas G_x^L and G_{xx}^L become singular as $(x, y, z) \to (\xi, \eta, 0)$, all but the last two integrands in (25) tend to zero and their integrals can thus be evaluated by Simpson's rule. Without such a desingularisation, the integrals cannot be determined accurately by numerical means.

The last two integrals can be performed using the appropriate antiderivative of F_y, namely

$$\begin{aligned}
F(x,y) &= \operatorname{sgn}(x)\operatorname{sgn}(y)\frac{k_0}{4\pi}\left[k_0|y|\left(\log\frac{k_0(r+|x|)}{4} + \gamma - 1\right)\right. \\
&\left. + 2k_0|x|\log k_0(r+|y|) - \frac{k_0|x||y|}{r+|x|}\right].
\end{aligned} \tag{27}$$

Since F_y is discontinuous at $y = 0$, care is required if $y < |b|$.

Although at points external to the region of applied pressure the derivatives of G^L are not singular, they can be large, especially if the field point is near the edge of the region. Then, the appropriate corrections are based on the pressure at the edge of the region. For example, for a point just ahead of the leading edge, one should replace $p(x, \eta)$ by $p(-a, \eta)$ and $p(x, y)$ by $p(-a, y)$. Again all integrands are well behaved, and the integrals can be determined accurately by Simpson's rule.

In this way, numerical integration can be used to evaluate the local-field x-velocity and hence the free-surface elevation at or near the region of applied pressure. Similar schemes can be used to determine the y and z components of the local-field velocity.

3 Code validation

We have verified our code by comparing with results published in the literature. Specifically, the wave resistance for a rectangular region of constant pressure agrees with that given by Newman and Poole [6] and others. Also, we have been able to reproduce the surface elevations shown in Huang and Wong's Figure 2 [3] and Cheng and Wellicome's Figure 3 [1]. The latter includes elevations for a pressure distribution that varies sinusoidally in the transverse direction. Finally, Lamb's two-dimensional result ([4], page 404) is reproduced along the centreline of rectangular constant-pressure distributions with large b/a ratios.

4 Conclusion

`SWPE-PD` uses the equations and techniques presented here. Armed with the above, an understanding of `SWPE` and the documentation within `SWPE-PD`, a suitably skilled person should be able to understand and, if necessary, modify the code to suit their specific requirements.

References

[1] Cheng, X. and Wellicome, J.F., *Study of planing hydrodynamics using strips of transversely variable pressure*, J. Ship Res. **38** (1994), pp. 30–

41.

[2] Doctors, L.J. and Day, A.H. *Wave-free river-based air-cushion vehicles.* Proc. Intnl. Conf. Hydrodynamics of High-Speed Craft Wake Wash and Motions Control, London England, November 2000.

[3] Huang, F.T. and Wong, K.K., *Disturbance induced by a pressure distribution moving over a free surface*, J. Ship Res., (1970) pp. 195–203.

[4] Lamb, H. *Hydrodynamics*, 5th edn, Cambridge 1936.

[5] Newman, J.N. *Evaluation of the wave-resistance Green function: Part 1 — The double integral.* J. Ship Res. **31** (1987) pp. 79–90.

[6] Newman, J.N. and Poole, F.A.P. *The wave resistance of a moving pressure distribution in a canal*, Schiffstechnik **9** (1962) pp. 1–6.

[7] Tuck, E.O., *Some accurate solutions of the lifting surface integral equation*, J. Austral. Math. Soc. Ser. B **35** (1993) pp. 127–144.

[8] Tuck, E.O., Collins, J.L. and Wells, W.H., *On ship waves and their spectra*, J. Ship Res. **15** (1971) pp. 11–21.

[9] Tuck, E.O., Lazauskas, L. and Scullen, D.C. *Sea Wave Pattern Evaluation, Part 1 report: Primary code and test results (surface vessels)*, Department of Applied Mathematics, The University of Adelaide, April 1999.

[10] Tuck, E.O., Scullen, D.C. and Lazauskas, L. *Sea Wave Pattern Evaluation, Part 2 report: Investigation of accuracy*, Department of Applied Mathematics, The University of Adelaide, May 1999.

[11] Tuck, E.O., Scullen, D.C. and Lazauskas, L. *Sea Wave Pattern Evaluation, Part 3 report: Near-field waves*, Scullen & Tuck Pty Ltd, Adelaide, January 2000.

[12] Tuck, E.O., Scullen, D.C. and Lazauskas, L. *Sea Wave Pattern Evaluation, Part 4 report: Extension to multihulls and finite depth*, Scullen & Tuck Pty Ltd, Adelaide, June 2000.

[13] Tuck, E.O., Scullen, D.C. and Lazauskas, L. *Sea Wave Pattern Evaluation, Part 5 report: Speed-up and squat*, Scullen & Tuck Pty Ltd, Adelaide, March 2001.

[14] Tuck, E.O., Scullen, D.C. and Lazauskas, L., *Ship-wave patterns in the spirit of Michell*. IUTAM Symposium on Free-Surface Waves, Birmingham, July 2000. Proc. ed. Y. Shikhmurzaev, Kluwer 2001.

[15] Wehausen, J.H. and Laitone, E.H., *Surface Waves*, in Handbuch der Physik, ed. Flügge S., Ch. 9. Springer-Verlag, 1962.